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ユニットメモリトレリス符号により構成されたブロック符号の誤り指数

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あらまし　ユニットメモリトレリス符号によって構成されたブロック符号について，ランダム符号化の立場から議論する．トレリス符号からブロック符号を得る方法には，(i) テールターミネーション（TT），(ii) ダイレクトトランケーション（DT），および (iii) テールバイティング（TB）の 3 つの方法がある．本論文では，ユニットメモリトレリス符号によって構成されたブロック符号について，上記 3 つの方法によるブロック符号の誤り指数を導出する．与えられたブランチ長さで，テールバイティングユニットメモリ（TB-UM）トレリス符号の誤り指数は，テールターミネーションユニットメモリ（TT-UM）および，ダイレクトトランケーションユニットメモリ（DT-UM）トレリス符号のそれらに比べ，通信路容量未満のすべてのレートにおいて大きいことを示す．また，復号の複雑さを考慮に入れると， TB-UM トレリス符号の復号誤り確率の上界は同一の復号の複雑さと低レートを除く同一のレートにおいて，通常のブロック符号のそれより小さいことを示す.

キーワード　ブロック符号，トレリス符号，テールバイティング，ユニットメモリ，誤り指数，復号計算量，最尤復号

Exponential Error Bounds for Block Codes Constructed by Unit Memory Trellis Codes

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Abstract　Block codes constructed by unit memory trellis codes are discussed from random coding arguments. There are three methods to obtain block codes from trellis codes, i.e., those of (i) Tail Termination (TT), (ii) Direct Truncation (DT), and (iii) Tail Biting (TB). In this paper, we discuss exponential error bounds for block codes constructed by the unit memory (UM) trellis codes based on the above three methods and error exponents for these codes are derived. For a given branch length, the error exponent of the tail biting unit memory (TB-UM) trellis codes is shown to be larger than those of the tail termination unit memory (TT-UM) and the direct truncation unit memory (DT-UM) trellis codes for all rates less than the capacity. Taking into account of the decoding complexity, the TB-UM trellis codes are also shown to have a smaller upper bound on the probability of decoding error than the ordinary block codes for the same rate except for low rates with the same decoding complexity.

Key words　Block codes, trellis codes, tail biting, unit memory, error exponents, decoding complexity, maximum–likelihood decoding

1. Introduction

It had been introduced to use unit memory (UM) convolutional codes as a byte oriented code [8]. In early 80's, bounds on free distances and error exponents of the UM trellis codes had been discussed [15]. The UM trellis codes had shown to have larger error exponents compared to ordinary trellis codes for low rates. Since we have already obtained powerful construction methods and efficient decoding algorithms for block codes, we can effectively use them as component codes of the UM trellis codes. Therefore, the UM trellis codes have a property combining the advantages of both block codes and trellis codes. Note, however, that decoding delay for the UM trellis codes has a probabilistic value. Hence sometimes, it is not tolerable to practical applications.

There are three methods to obtain block codes from trellis codes, i.e., those of (i) Tail Termination (TT), (ii) Direct Truncation (DT), and (iii) Tail Biting (TB). Especially, (iii) tail biting (TB) trellis codes [10] are known to be one of the most powerful codes for converting trellis codes into block codes with no loss in rates. Since the TB trellis codes require an intolerable increase in the decoding complexity, much efforts have been devoted to the studies on suboptimum decoding algorithms [1], [10] or efficient maximum-likelihood decoding algorithms [11], [14], [16]. Unfortunately, however, the decoding complexity of the latter algorithms in worst case are the same as that of the complete maximum-likelihood decoding algorithm, although it is asymptotically the same as that of the Viterbi algorithm when the signal to noise ratio becomes large [16].

On the other hand, a coding theorem obtained by classical random coding arguments gives us simple and elegant results on coding schemes, although it states only an existence of a code. Random coding arguments can demonstrate the essential mechanism on coding systems. Since we require complete maximum-likelihood decoding (MLD), the relationship between the probability of decoding error $\Pr(\mathcal{E})$ and the decoding complexity $G(N)$ at a given rate r, or R, can be made clear, where N is the code length. It should be noted that the coding theorem can only suggest the behavior of the code ensemble, hence it is not useful enough to design a practical code.

In this paper, we discuss the performance of the block codes constructed by the UM trellis codes based on the above three methods, i.e., the Tail Termination UM (TT-UM) trellis codes, the Direct Truncation UM (DT-UM) trellis codes, and the Tail Biting UM (TB-UM) trellis codes. For a given branch length, the error exponent of the TB-UM trellis codes is shown to be larger than those of the TT-UM and the DT-UM trellis codes for all rates less than the capacity. Taking into account of the decoding complexity $G(N)$, the former is also shown to have a smaller upper bound on the $\Pr(\mathcal{E})$ than the ordinary block codes for the same $r = R$ except for low rates with the same $G(N)$.

Throughout this paper, assuming a discrete memoryless channel with capacity C, we discuss the lower bounds on the reliability function $E(\cdot)$ for block codes and $e(\cdot)$ for trellis codes, and asymptotic decoding complexity G measured by the computational work [12]. The probability of decoding error is denoted by $\Pr(\mathcal{E})$, the rate, r or R, the code length, N, and the decoding complexity, $G(N)$.

In section 2, we briefly review on the error exponents of the block codes and the trellis code as preliminaries. Section 3 shows the results of the UM trellis codes [15]. We derive error exponents and decoding complexity for the TT-UM trellis codes, the DT-UM trellis codes, and the TB-UM trellis codes in section 4 [1]. Section 5 discusses the results on the exponential error bounds and asymptotical decoding complexity. Section 6 is concluding remarks of this paper.

2. Preliminaries

2.1 Block codes

Let an (N, K) block codes over $GF(q)$ be an ordinary block code of length N, number of information symbols K, and rate R, where

$$R = (K/N)\ln q \quad (K \leqq N). \quad \text{[nats/symbol]} \tag{1}$$

From random coding arguments for an ordinary block code, there exists a block code of length N and rate R for which the probability of decoding error $\Pr(\mathcal{E})$ and the decoding complexity $G(N)$ satisfy

$$\Pr(\mathcal{E}) \leqq \exp[-NE(R)] \quad (0 \leqq R < C), \tag{2}$$

and

$$G(N) \sim N\exp[NR], \tag{3}$$

where $E(\cdot)$ is (a lower bound on) the block code exponent [5], and the symbol "$\sim$" indicates asymptotic equality.

2.2 Trellis codes

Let a (u, v, b) trellis code over $GF(q)$ be a code of branch length u, branch constraint length v, yielding b channel symbols per branch, and rate r, which satisfies

$$r = (1/b)\ln q. \quad \text{[nats/symbol]} \tag{4}$$

The probability of decoding error $\Pr(\mathcal{E})$ and the decoding complexity $G(v)$ satisfy [5]

$$\begin{aligned} \Pr(\mathcal{E}) &\leqq uK_1\exp[-vbE_0(\rho)] \qquad (0 \leqq \rho \leqq 1) \quad (5)\\ &\leqq \exp\{-vb[e(r) - o(1)]\} \quad (0 \leqq r < C), \quad (6)\end{aligned}$$

and

$$G(v) \sim u^2 q^v, \tag{7}$$

where K_1 is a constant independent of v, $o(1) \to 0$ $(v \to \infty)$, and $e(\cdot)$ is (a lower bound on) the trellis code exponent [5] given by

$$e(r) = \begin{cases} E_0(1) & 0 \leqq r \leqq R_{\text{comp}} \\ E_0(\rho_r) & R_{\text{comp}} < r = E_0(\rho_r)/\rho_r < C, \end{cases} \tag{8}$$

where $E_0(\rho)$ is the Gallager's function, and $R_{\text{comp}} = E_0(1)$ is the computational cut-off rate of the channel. Note that the following relation holds between $E(R)$ and $e(r)$:

$$E(R) = \max_{0<\mu\leqq 1}(1-\mu)e(R/\mu), \tag{9}$$

which is called the concatenation construction [5]. Similarly, letting

$$\theta = v/u, \tag{10}$$

the following equation also holds [5]:

1) : This paper describes the detailed derivations or the proof of [6].

$$e(r) = \min_{0<\theta\leqq 1} (1/\theta)E[(1-\theta)r], \quad (11)$$

which is called the inverse concatenation construction [5]. The block code exponent $E(R)$, and the trellis code exponent $e(r)$ are shown in Figure 1 for a very noisy channel.

3. Unit memory (UM) trellis codes

For the (u, v, b) conventional trellis code discussed in the previous section, letting $v = 1$ and introducing an (n_0, k_0) block code to each branch as a component code, we have a (u, n_0, k_0) unit memory (UM) trellis code. Note that we let $n_0 \to \infty$ for the UM trellis codes, while $v \to \infty$ for the conventional trellis codes.

Let a (u, n_0, k_0) unit memory (UM) trellis code over $GF(q)$ be a code of branch length u, length of component code n_0, number of information symbols of component code k_0, and rate r, which satisfies

$$r = (k_0/n_0)\ln q. \quad \text{[nats/symbol]} \quad (12)$$

[Lemma 1] (Thommesen & Justesen [15]) There exists a (u, n_0, k_0) unit memory (UM) trellis code for which the probability of decoding error $\Pr(\mathcal{E})$ and the decoding complexity $G(n_0)$ satisfy

$$\Pr(\mathcal{E}) \leqq \exp[-n_0 e_{\mathrm{UM}}(r)] \quad (0 \leqq r < C), \quad (13)$$

and [2)]

$$G(n_0) \sim u^2 n_0 q^{2k_0} = u^2 n_0 \exp[2n_0 r], \quad (14)$$

where

$$e_{\mathrm{UM}}(r) = \begin{cases} 2E(r/2) & (0 \leqq r \leqq R_{\mathrm{comp}}) \\ e(r) & (R_{\mathrm{comp}} < r < C), \end{cases} \quad (15)$$

holds. □

We then easily have [15]

$$e_{\mathrm{UM}}(r) = 2E(r/2) > e(r) \quad (0 \leqq r < R_{\mathrm{comp}}). \quad (16)$$

4. Block codes constructed by unit memory (UM) trellis codes

Let us consider a (u, n_0, k_0) unit memory (UM) trellis code, and we let this code be converted to a block code by using the methods described above. Since we shall compare the conventional (N, K) block code to the block codes constructed by the UM trellis codes, the code length N is chosen as:

$$N = u_0 n_0, \quad (17)$$

denoting

$$0 \leqq \theta_0 = 1/u_0 \leqq 1/2, \quad (18)$$

where u_0 is the branch length and is an integer such that $u_0 = 2, 3, \cdots$.

4.1 Tail termination unit memory (TT-UM) trellis codes

Letting a (u, n_0, k_0) unit memory code be terminated at branch u_0, we have a (u_0, n_0, k_0) tail termination unit memory (TT-UM) trellis code over $GF(q)$ of length $N = u_0 n_0$, and rate R. We then easily have the following lemma [5], where actual rate $R = (1-\theta_0)r$, and $r = (k_0/n_0)\ln q$.

[Lemma 2] The probability of decoding error $\Pr(\mathcal{E})$ and the decoding complexity $G(N)$ for the TT-UM trellis code are given by

$$\Pr(\mathcal{E}) \leqq \exp[-N E_{\text{TT-UM}}(R)] \quad (0 \leqq R < C), \quad (19)$$

where

$$E_{\text{TT-UM}}(R) = \tilde{E}(R), \quad (20)$$

$$\tilde{E}(R) = \theta_0 e_{\mathrm{UM}}[R/(1-\theta_0)] \quad (\theta_0 = 1/u_0, u_0 = 2, 3, 4, \cdots), \quad (21)$$

and

$$G(N) \sim u_0 N q^{2k_0} = u_0 N \exp[2N\theta_0 r]. \quad (22)$$

□

If θ_0 takes real values, then (21) gives $E(R)$ by the concatenation construction [5] of (9) replaced by $\nu = 1 - \theta_0$. However, $\theta_0 = 1/u_0, u_0 = 2, 3, 4, \cdots$, the rhs of (21) is consisted by the set of straight lines for discrete values of θ_0. Hence the following relation holds:

$$\tilde{E}(R) \leqq E(R), \quad (23)$$

where the lhs of (23) are given by the set of tangent lines to $E(R)$ which is the upper envelop of the straight lines.

4.2 Direct truncation unit memory (DT-UM) trellis codes

Similarly, truncation at branch u_0 of a (u, n_0, k_0) unit memory trellis code gives a (u_0, n_0, k_0) direct truncation unit memory (DT-UM) trellis code over $GF(q)$ of length $N = u_0 n_0$, and rate r. We then have the following lemma, where there is no loss in rate.

[Lemma 3] For the DT-UM trellis code, $\Pr(\mathcal{E})$, and $G(N)$ are given by

$$\Pr(\mathcal{E}) \leqq \exp[-N e_{\text{DT-UM}}(r)] \quad (0 \leqq r < C), \quad (24)$$

where

$$E_{\text{DT-UM}}(r) = E(r), \quad (25)$$

and

$$G(N) \sim u_0 N q^{2k_0} = u_0 N \exp[2N\theta_0 r], \quad (26)$$

where $N = u_0 n_0$, and $r = (k_0/n_0)\ln q$. □

(Proof) See Appendix A.

4.3 Tail biting unit memory (TB-UM) trellis codes

Again consider a (u, n_0, k_0) unit memory (UM) trellis code, and we let this code be converted to a block code by using tail biting techniques.

Let $\boldsymbol{w} \in \mathcal{X}^{u_0 k_0}$ be a message sequence of (branch) length u_0, where $\mathcal{X}$ is the channel input alphabet. Each node is composed of an (n_0, k_0) block component code and rewrite w as

$$\boldsymbol{w} = (\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_{u_0}). \quad (27)$$

2) : The number of the states of the Viterbi decoder is q^{k_0}, and at each state it compares q^{k_0} survivors. The decoder repeats it u times, where the length of the register is un_0. Then the over-all decoding complexity of the UM trellis code is given by (14).

[Encoding Procedure]

(i) Initialize the encoder by inputting the last k_0 information (branch) symbols $\boldsymbol{w}_{u_0}$ of $u_0 k_0$ information (branch) symbols $\boldsymbol{w}$, and disregard the output of the encoder.

(ii) Input all $u_0 k_0$ information symbols $\boldsymbol{w}$ into the encoder, and output the codeword $\boldsymbol{x} \in \mathcal{X}^N$ of length $N = u_0 n_0$ in channel symbols, where rate $r = (k_0/n_0)\ln q$. □

The resultant code is a (u_0, n_0, k_0) tail biting unit memory (TB-UM) trellis code, and the rate of this code is the same as that of the UM trellis code. Hence there is no loss in rate, in contrast to the TT-UM trellis code whose rate reduces to $R = (1-\theta_0)r$.

[Theorem 1] There exists a (u_0, n_0, k_0) tail biting unit memory (TB-UM) trellis code for which the probability of decoding error $\Pr(\mathcal{E})$ and the decoding complexity $G(N)$ satisfy

$$\Pr(\mathcal{E}) \leqq \exp[-N E_{\text{TB-UM}}(r)] \quad (0 \leqq r < C), \qquad (28)$$

where

$$E_{\text{TB-UM}}(r) = \min_{\theta_0 = 1/u_0, u_0 = 2,3,4,\cdots} \{\theta_0 e_{\text{UM}}(r), E[(1-\theta_0)r]\}, \qquad (29)$$

and

$$G(N) \sim u_0 N q^{3k_0} = u_0 N \exp[3N\theta_0 r], \qquad (30)$$

where $N = u_0 n_0$ and $r = (k_0/n_0)\ln q$. □

(Proof) See Appendix B [3].

5. Discussions

5.1 Exponential error bounds

We have derived error exponents as given by (20), (25), and (29) for the TT-UM, the DT-UM, and the BT-UM trellis codes, respectively. As the summary, we have the following corollary from (23).

[Corollary 1] For the same rate $r = R$, and a given θ_0 $(0 \leqq \theta_0 \leqq 1/2)$, the following relation holds:

$$E_{\text{TT-UM}}(R) \leqq E_{\text{DT-UM}}(r) \leqq E_{\text{TB-UM}}(r). \qquad (31)$$

□

Finally, we give computational results for a very noisy channel.

[Example 1] Over a very noisy channel (VNC), the error exponent $E_{\text{TB-UM}}(r)$ for the TB-UM trellis codes is depicted in Figure 1 together with that $E_{\text{TT-UM}}(R)$ for the TT-UM trellis code. As stated in Corollary 1, for a given u_0, $E_{\text{TT-UM}}(R) \leqq E_{\text{TB-UM}}(r)$ holds for all rates $r = R$. Since $\theta_0 = 1/2, 1/3, \cdots$, $E_{\text{TT-UM}}(R)$ is given by the set of straight lines, and it is close to the curve $E(R)$ as its tangent lines, hence $E_{\text{TT-UM}}(R) \leqq E(R)$ always holds. Note that by the result of numerical computation, the second term of the rhs of (29) does not affect $E_{\text{TB-UM}}(r)$ for all rates over the VNC. □

3) : To prove Theorem 1, a decoding procedure is stated also in Appendix B.

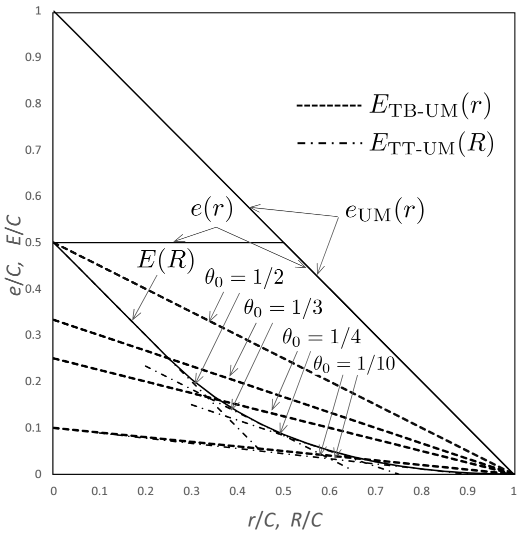

Figure 1 Error exponents for a very noisy channel $(0 \leqq \theta_0 \leqq 1/2)$.

5.2 Asymptotic decoding complexity

Using (22), (26), and (30), we shall discuss the decoding complexity of the block codes constructed by the UM trellis code based on the three methods. The probability of decoding error $\Pr(\mathcal{E})$ can be rewritten in terms of the decoding complexity G as shown in the following.

[Corollary 2] The following inequalities hold for a given θ_0 $(0 \leqq \theta_0 \leqq 1/2)$:

For the TT-UM trellis codes

$$\Pr(\mathcal{E}) \lesssim G^{-\frac{(1-\theta_0)E(R)}{2\theta_0 R}}, \qquad (32)$$

for the DT-UM trellis codes

$$\Pr(\mathcal{E}) \lesssim G^{-\frac{E(R)}{2\theta_0 R}}, \qquad (33)$$

and for the TB-UM trellis codes

$$\Pr(\mathcal{E}) \lesssim G^{-\frac{e_{\text{UM}}(r)}{3r}}. \qquad (34)$$

□

(Proof) See Appendix C.

Comparing the ordinary block code and the TB-UM trellis code, we have the following interesting theorem.

[Theorem 2] For the same decoding complexity $G(N)$ and the same rate $r = R$, for all rates except for low rates, the upper bound on the probability of decoding error $\Pr(\mathcal{E})$ for the TB-UM trellis code is asymptotically smaller than that for the ordinary block code. □

(Proof) See Appendix D.

This theorem suggests us that we can attain smaller $\Pr(\mathcal{E})$ for the TB-UM trellis code than that for the block code, for all rates except for low rates [4], although the decoding complexity G of the former grows exponentially with $3N$, while that of the latter, with only N. Note that, however, the $\Pr(\mathcal{E})$ decreases only algebraically with G.

4) : Over VNC, it is valid for $C/4 \leqq r = R < C$, which is also true for tail biting trellis codes [7].

6. Concluding remarks

We have shown that the upper bound on the probability of decoding error of the block code is improved by using the tail biting unit memory (TB-UM) trellis codes at all rates less than the capacity. It is also true for all rates except for low rates, in the case when taking into account of the decoding complexity, even if the decoding complexity of the TB-UM trellis code is in the cube order of that of the block code.

In this paper, we have discussed on block codes by applying only the UM trellis codes. The performance of block codes obtained by original trellis codes, and comparison of them with the TB-UM trellis codes discussed in this paper are remained as future works [7].

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Appendix

Appendix A

Followed by (126) of [5], we have

$$\Pr(\mathcal{E}) \leqq q^{u_0 k_0 \rho} \exp[-u_0 n_0 E_0(\rho)] \leqq \exp[-NE(r)]. \quad (\mathrm{A{\cdot}1})$$

Appendix B

First, we shall give a complete maximum-likelihood decoding procedure for the TB-UM trellis code. The over-all decoder consists of q^{k_0} Viterbi decoders called sub-trellis decoders, each of which starts at the state s_i, and ends at (the same) state s_i for the i-th sub-trellis, where $i = 1, 2, \cdots, q^{k_0}$ as shown in Figure A· 1. Let the code words of the i-th sub-trellis be represented by $\boldsymbol{x}_{i,j}$, where $j = 1, 2, \cdots, q^{(u_0-1)k_0}$, and the received sequence of length $N = u_0 n_0$ be denoted by $\boldsymbol{y} \in \mathcal{Y}^N$, where $\mathcal{Y}$ is the channel output alphabet.

[Decoding Procedure]

(i) Decode $\boldsymbol{y}$ by the i-th Viterbi decoder, and output the decoded code word $\hat{\boldsymbol{x}}_i$, i.e.,

$$\hat{\boldsymbol{x}}_i = \arg \max_{j=1,2,\cdots,q^{(u_0-1)k_0}} \Pr(\boldsymbol{y}|\boldsymbol{x}_{i,j}). \quad (\mathrm{A{\cdot}2})$$

(ii) Compute [5)] and output the over-all decoded code word $\hat{\boldsymbol{x}}$, i.e.,

$$\hat{\boldsymbol{x}} = \arg \max_{j=1,2,\cdots,q^{k_0}} \Pr(\boldsymbol{y}|\hat{\boldsymbol{x}}_i). \quad (\mathrm{A{\cdot}3})$$

□

Next, we shall derive the error exponent of the TB-UM trellis code. Without loss of generality, we let the true path be $\boldsymbol{x}_1^* = 0^N$, i.e., $u_0 n_0$-tuple of 0s start at s_1 (and end at s_1) (See Figure A· 1). Let us define the error event $\mathcal{E}_1, \mathcal{E}_2$, and $\mathcal{E}_3$ as follows:

$\mathcal{E}_1$: The error event of the 1-st sub-trellis which contains the true path $\boldsymbol{x}_1^*$, where the survivors $\boldsymbol{x}_{1,j}$ remerge with $\boldsymbol{x}_1^*$ until the u_0-th branch level (which does not contain the event of $\mathcal{E}_3$).

5) : Note that by (A·3), we can obtain the over-all decoded code word of maximum-likelihood decoding (MLD) for total $q^{u_0 k_0}$ code words.

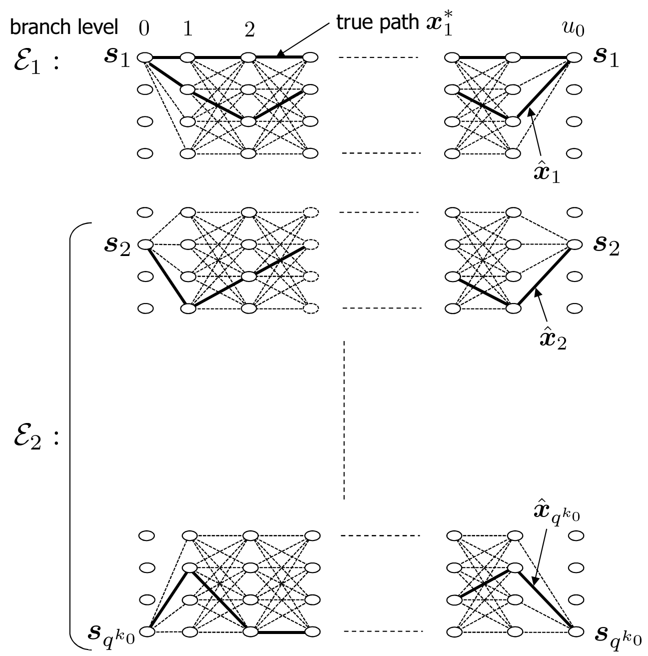

Figure A· 1 Trellis diagrams of a TB-UM trellis code ($q = 2, k_0 = 2$).

$\mathcal{E}_2$: The error events of the all except for the 1-st (i.e., 2-nd, 3-rd, $\cdots$, q^{k_0}-th) sub-trellises which do not contain the true path, and also do not contain the event of $\mathcal{E}_3$.

$\mathcal{E}_3$: The error event for which the all survivors diverge at the 0-th branch level, and never remerge with $\boldsymbol{x}_1^*$ until the u_0-th branch level in the all sub-trellises.

Then the probability of decoding error $\Pr(\mathcal{E}_1)$ for the 1-st sub-trellis is upper bounded by that for the UM trellis code, since the latter contains the extra error events which never remerge with $\boldsymbol{x}_1^*$ until the u_0-th branch level. Then we have

$$\begin{aligned}\Pr(\mathcal{E}_1) &\leqq \exp[-n_0 e_{\mathrm{UM}}(r)] \\ &= \exp[-N\theta_0 e_{\mathrm{UM}}(r)], \qquad \text{(A·4)}\end{aligned}$$

where an error event begins at any time. While assuming the (-1)-th branch level (starting at s_1 at the (-1)-th branch level), and shifting the branch level by (-1), we see that the probability of decoding error $\Pr(\mathcal{E}_2)$ within the sub-trellises starting at s_i and ending at s_i ($i \neq 1, i = 2, \cdots, q^{k_0}$) is bounded by that of $\Pr(\mathcal{E}_1)$. Next, we intend to derive $\Pr(\mathcal{E}_3)$ by introducing the probability of list decoding error $\Pr(\mathcal{L})$, where an event of list decoding error $\mathcal{L}$ occurs when the true path $\boldsymbol{x}_1^*$ is not on the over-all output list $\hat{\boldsymbol{x}}_i$ ($i = 1, 2, \cdots, q^{k_0}$) [2], [4], [13]. If such an event occurs, the output list never contain the true path i.e., the all decoded outputs on the list are in error. The $\Pr(\mathcal{L})$ can be derived as follows: If [6])

$$\Pr(\boldsymbol{y}|\boldsymbol{x}_1^*) \leqq \Pr(\boldsymbol{y}|\hat{\boldsymbol{x}}_i), \quad \text{for at least } |\mathcal{L}| \text{ distinct } \hat{\boldsymbol{x}}_i \quad (i = 1, 2, \cdots, |\mathcal{L}|) \qquad \text{(A·5)}$$

holds, where we choose the list size $|\mathcal{L}| = q^{k_0}$, then the probability of list decoding error $\Pr(\mathcal{L})$ is given by [2], [4], [13]

$$\Pr(\mathcal{L}) \leqq \exp[-NE(r')], \qquad \text{(A·6)}$$

where

$$r' = (1/N)\ln(M/|\mathcal{L}|) = (1-\theta_0)r, \qquad \text{(A·7)}$$

and $M = q^{u_0 k_0} = \exp[Nr]$, and $|\mathcal{L}| = q^{k_0} = \exp[N\theta_0 r]$. Consequently, we have the over-all probability of decoding error $\Pr(\mathcal{E})$ is derived as

$$\begin{aligned}\Pr(\mathcal{E}) &\leqq \Pr(\mathcal{E}_1) + \Pr(\mathcal{E}_2) + \Pr(\mathcal{E}_3) \\ &\leqq 2\Pr(\mathcal{E}_1) + \Pr(\mathcal{L}) \\ &\leqq \exp\{-N[\theta_0 e_{\mathrm{UM}}(r) - o(1)]\} \\ &\quad + \exp\{-NE[(1-\theta_0)r]\}. \qquad \text{(A·8)}\end{aligned}$$

Ignoring $o(1) = \ln 2/N \to 0$ as $N \to \infty$, we have (29).

Finally, we discuss the decoding complexity of the TB-UM trellis code. The number of the states of the single sub-trellis with the initial state s_i ($i = 1, 2, \cdots, q^{k_0}$) is q^{k_0}, and at each state (node) the Viterbi algorithm compares q^{k_0} survivors. The decoder repeats it u_0 times where the length of the register is N. The number of such subtrellises is q^{k_0}. Thus the over-all complexity is uNq^{3k_0}, which leads (30), completing the proof.

[Lemma 4] Let $\boldsymbol{y}$ be decoded into $\hat{\boldsymbol{x}}$, and $\hat{\boldsymbol{x}}$ never merge with $\boldsymbol{x}_1^*$ until the u_0-th branch level, then we let such $\boldsymbol{y}$ be denoted by $\boldsymbol{y} \in Y_{u_0+1}$. If $\boldsymbol{y} \in Y_{u_0+1}$, then there are $\hat{\boldsymbol{x}}_i$s which satisfy (A·5). □

(Proof) Assume the best component code whose average probability of decoding error p_e is least and satisfies [3]

$$p_e = \exp\{-n_0[E(r) + o(1)]\}. \qquad \text{(A·9)}$$

When $\boldsymbol{y} \in Y_{u_0+1}$, the transmitted true path $\boldsymbol{x}_1^*$ is received as $\boldsymbol{y}$ such that all the component codes are in error, hence

$$\begin{aligned}\Pr(\boldsymbol{y}|\boldsymbol{x}_1^*) &\leqq [p_e]^{u_0}/[\exp(u_0 n_0 r) - 1] \\ &\leqq \frac{\exp\{-u_0 n_0[E(r) + o(1)]\}}{\exp(u_0 n_0 r) - 1}, \qquad \text{(A·10)}\end{aligned}$$

holds, using the Viterbi algorithm which performs MLD. There is the MLD path $\hat{\boldsymbol{x}}$ such that $\boldsymbol{y}$ lies in the decoding region of $\hat{\boldsymbol{x}}$, which satisfies

$$\begin{aligned}\Pr(\boldsymbol{y}|\hat{\boldsymbol{x}}) &\geqq 1 - [p_e]^{u_0} \\ &\geqq 1 - \exp\{-u_0 n_0[E(r) + o(1)]\}, \qquad \text{(A·11)}\end{aligned}$$

for some $\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}_i$, and for $i' \neq i$

$$\begin{aligned}\Pr(\boldsymbol{y}|\hat{\boldsymbol{x}}_{i'}) &\geqq (1 - \exp\{-(u_0 - 1)n_0[E(r) + o(1)]\}) \\ &\quad \cdot \frac{\exp\{-n_0[E(r) + o(1)]\}}{\exp(n_0 r) - 1} \\ &\sim \frac{\exp\{-n_0[E(r) + o(1)]\}}{\exp(n_0 r) - 1}. \qquad \text{(A·12)}\end{aligned}$$

Equation (A·10) and (A·12) lead (A·5) for specified $\boldsymbol{y} \in Y_{u_0+1}$.

Appendix C

From (22), we have

$$G(N) \sim u_0 N q^{2k_0} = \exp\{2N[1 + o(1)]\theta_0 r\}, \qquad \text{(A·13)}$$

where $o(1) = (1/2N)\ln u_0 N \to 0$ as $N \to \infty$. Then N is represented by

$$N \sim \ln G/2\theta_0 r. \qquad \text{(A·14)}$$

Substitution of (A·14) into (20) gives (32), where we have used $R = (1-\theta_0)r$. Similarly, (24) and (26) give (33), and also (28) and (30) give (34). In above derivations, the all terms $o(1)$ are ignored, since we are interested only in asymptotics.

Appendix D

From (2) and (3), we have for an ordinary block code

$$N \sim \ln G/R, \qquad \text{(A·15)}$$

Substitution of (A·15) into (2) gives

$$\Pr(\mathcal{E}) \lesssim G^{-E(R)/R}. \qquad \text{(A·16)}$$

If $E(R) \leqq e_{\mathrm{UM}}(r)/3$ holds, we have

$$E(R)/R \leqq e_{\mathrm{UM}}(r)/3r. \qquad \text{(A·17)}$$

Actually, over VNC, (A·17) holes for rate $C/4 \leqq r = R < C$. This leads Theorem 2.

6) : See the following Lemma 4.